Density of proper delay times in chaotic and integrable quantum billiards

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We calculate the density $P(\tau)$ of the eigenvalues of the Wigner-Smith time delay matrix for two-dimensional rectangular and circular billiards with one opening. For long times, the density of these so-called "proper delay times" decays algebraically, in contradistinction to chaotic quantum billiards for which $P(\tau)$ exhibits a long-time cutoff.

In classical mechanics, the length of time that a single particle remains within a certain region of space is a uniquely defined quantity. In quantum mechanics, however, the concept of "time spent in a region" is not well defined, because there is no such thing as a Hermitian "time-delay operator" [1]. For scattering in one dimension, Wigner has constructed a quantum-mechanical delay time in terms of the energy derivative of the phase shift acquired upon scattering from the region or potential of interest [2]. His concept was generalized by Smith [3], who introduced a matrix of delay times,

$$Q = -i\hbar S^{\dagger} \frac{\partial S}{\partial \varepsilon}, \qquad (1)$$

where S is the scattering matrix, ε is the energy of the incident particle, and $\hbar$ is Planck's constant. In addition to being relevant for the retardation of a wave packet, the Wigner-Smith time-delay matrix Q has been shown to be related to the capacitance [4,5], thermopower [6], and, indirectly, to parametric conductance derivatives [7] and quantum pumping [8–10].

In this article, we consider the eigenvalues of the Wigner-Smith time-delay matrix for scattering from a cavity. The eigenvalues τ_n, $1, \ldots, N$ of Q are known as "proper delay times" [3]. A cavity coupled to the outside world via a waveguide with N propagating channels at energy ε is characterized by N proper delay times. In the semiclassical limit of large N, the system is described by the density $P(\tau)$ of delay times. Our aim is to compare $P(\tau)$ for cases where the classical dynamics of the cavity is chaotic or integrable. On the level of classical delay times, it is well known that the delay time distributions are different for these two cases: For an integrable cavity, $P_{\text{class}}(\tau) \propto \tau^{-\gamma}$ has algebraic tails for large τ, whereas P_{class} decays exponentially for large τ if the cavity has chaotic classical dynamics [11,12]. Simple arguments based on the proximity of initial conditions to trapped periodic orbits set $\gamma \lesssim 3$ for the two-dimensional square and circular billiards, the two examples of integrable cavities we consider here.

The density $P(\tau)$ of proper delay times for a chaotic cavity was calculated by Frahm, Beenakker, and one of the authors [13,14] using random matrix theory [15]. It was found that $P(\tau)$ has a finite support,

$$P(\tau) = \begin{cases} \dfrac{1}{2\pi\tau^2}\sqrt{(\tau_+ - \tau)(\tau - \tau_-)}, & \tau_- < \tau < \tau_+ \\ 0 & \text{otherwise,} \end{cases} \qquad (2)$$

where $\tau_{\pm} = \bar{\tau}(3 \pm \sqrt{8})$, $\bar{\tau}$ being the average delay time. The density of proper delay times for an integrable cavity is markedly different, as we now show by consideration of the rectangular and circular billiards in two dimensions.

To find the density of proper delay times for the integrable billiards, we solve the Schrödinger equation in the billiard and the leads separately, and match the wave functions at the billiard–lead interface [16]. By repeating this process for all possible incident modes in the lead, a system of linear equations is formed from which the scattering matrix S, its derivative $\partial S/\partial \varepsilon$, and, hence, the time-delay matrix Q can be calculated. The billiards and the leads are shown to scale in the insets of Figs. 1 and 2. To improve our statistics, we have performed an average over the position of the lead and small area-preserving fluctuations of the billiard aspect ratio for the rectangular billiard [17], and the small fluctuations of the energy ε for the circular billiard. Plots of the ensemble-averaged $P(\tau)$ for these two types of billiard are shown in Figs. 1 and 2, respectively. We have studied the long-time asymptotics through the integrated density $\int_{\tau} P(\tau') d\tau'$; see the insets of Figs. 1 and 2. For large N, the averaged density is representative of the density $P(\tau)$ of a particular billiard,
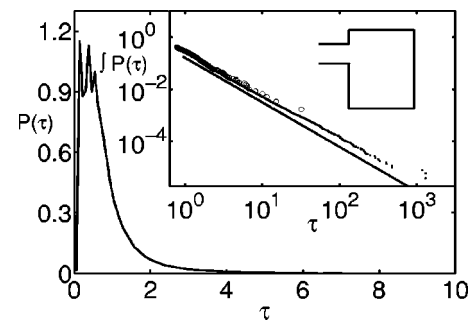

FIG. 1. Density P of proper delay times for a rectangular billiard (shown in the inset). The density P is normalized to unity, $\int P(\tau) d\tau = 1$, and the delay times τ are measured in units of their average $\bar{\tau}$. The inset shows the long-time tail of the cumulative distribution $\int_{\tau} P(\tau') d\tau'$, together with a (displaced) linear fit with a slope of -1.70. The delay times were taken for an incident wave with energy near the 17 600 level of the billiard, at which the contact has $N=41$ propagating modes, and the density shown was obtained after averaging over small size-preserving variations of the shape of the billiard and the position of the lead. The circles denote the cumulative density for a single realization near the 3.5×10^6 energy level in the billiard with $N=585$ and a fit slope of -1.66.
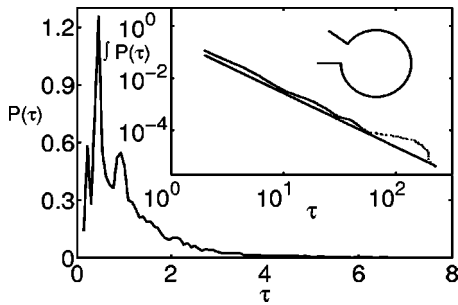
FIG. 2. Density of proper delay times for the circle billiard (shown in the inset) and the corresponding long-time tail of the cumulative density $\int_{\tau} P(\tau')d\tau'$. The distribution shown is obtained after averaging over small variations of the energy around the 2250 energy level of the billiard, for which there were 18 conducting modes in the point contact.

as is shown in Fig. 1, where we compare the tail of the averaged $P(\tau)$ with the density of proper delay times for a single realization.

As can be seen from Figs. 1 and 2, the density $P(\tau)$ was found to decay algebraically for large τ for both the rectangular and circular billiards. In the rectangular billiard, the exponent of the decay was estimated as $\gamma = 2.6$, independent of energy in the inspected energy range of $10^3 < \varepsilon < 2 \times 10^4$ (energy is measured in units such that the level spacing in the closed billiard is 1), giving a range in the number of propagating modes in the contact ranging from $N = 9$ to 41. In the circular billiard we inspected energies in the range of $10^3 < \varepsilon < 3 \times 10^3$ (corresponding to N ranging from 14 to 20), and found that P decayed with an algebraic tail with exponent $\gamma = 3.05$, again independent of energy within our accuracy. (The result that γ is larger in the circular case could be attributed to rounding errors in calculating Bessel functions in the large N regime of interest, which introduce small random fluctuations that mimic a small randomness in the potential, thereby suppressing long dwell times like in the chaotic regime. The jaggedness of the small τ distribution in the circular case is due to the restricted avenues of averaging available compared to in the rectangular case.) The classical distribution of delay times decays $\propto \tau^{-\gamma}$ with $\gamma \approx 3$ in both cases [18]. Hence, we conclude that the density of proper delay times decays algebraically for the integrable billiard we studied, with a power that is close but (for our observations) not precisely equal to the classical power $\gamma = 3$. Our results may be compared to one-dimensional quasiperiodic systems [19] which also exhibit an algebraic decay of the delay time distribution.

We wish to point out that the difference in the *density* of delay times for chaotic and integrable cavities reported here is something quite remarkable. Although it is known that many quantum properties are different for cavities with chaotic and integrable classical dynamics, these differences usually pertain to the statistical fluctuations, described by correlation functions, or show up in small quantum interference corrections to a (semi)classical background [11,20], but not in the (ensemble averaged) densities themselves. The only

exception known to the authors is the density of states $\rho(\varepsilon)$ in a so-called "Andreev quantum billiard," an "electron billiard" that is connected to a superconducting point contact. In this case, electrons that exit the cavity and impinge on the superconductor interface are reflected as holes and vice versa. As a result of this special reflection process, known as Andreev reflection [21], $\rho(\varepsilon)$ is singular around the Fermi energy $\varepsilon = 0$: The density of states has a gap for a cavity with chaotic classical dynamics, while for an integrable cavity, $\rho(\varepsilon)$ decays algebraically as $\varepsilon \to 0$ [22]. In fact, this feature can be connected to the difference for the tail of the density of proper delay times $P(\tau)$ reported here via the following heuristic argument: As shown in Ref. [22], the Andreev quantum billiard has an eigenstate at precisely energy ε if the matrix product $S(\varepsilon)S^{\dagger}(-\varepsilon)$ has an eigenvalue of -1. Expanding S around $\varepsilon = 0$ gives

$$S(\varepsilon)S^{\dagger}(-\varepsilon) \approx e^{2i\varepsilon Q/\hbar}. \tag{3}$$

With this approximation, the condition for eigenstates in a cavity coupled to a superconductor simplifies to [23]

$$e^{2i\varepsilon \tau_n/\hbar} = -1, \tag{4}$$

where τ_n is a proper delay time (eigenvalue of Q). Equation (4), viewed as a constraint on the product $\varepsilon \tau_n$, connects $\rho(\varepsilon)$ at small ε to $P(\tau)$ at large τ. The fact that the density of states is gapped for the chaotic Andreev quantum billiard can then be understood as following from the absence of a large-time tail of $P(\tau)$, [cf. Eq. (2)] whereas the algebraic vanishing of $\rho(\varepsilon)$ near zero energy for the integrable Andreev quantum billiard is seen to be related to the algebraic tail of the density of delay times in that case [24]. We note, however, that the approximation, Eq. (3), cannot be used for a quantitative estimate of the density of states, since it becomes unreliable for energies of $\varepsilon \tau \sim \hbar$, which is precisely where the first eigenstates are expected to appear.

While we are not aware of a method by which to directly measure the density of proper delay times for an electronic system, a direct measurement of $P(\tau)$ would be possible using the scattering of electromagnetic waves (microwave radiation) from a metal cavity [25–28]. With a suitable choice of basis, S and Q may be simultaneously diagonalized [13]. If the incoming electromagnetic radiation is in a plane wave state that corresponds to one of these basis vectors and is slowly modulated in intensity, the ac modulation of the outgoing waves will be delayed by the proper delay time τ_n corresponding to the incoming wave mode [3]. In this way, the qualitative difference between chaotic and integrable cavities should be readily accessible from the tail of the density of proper delay times, without further ensemble averaging.

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